

The BCS-BEC crossover induced by a shallow band: Pushing standard superconductivity types apart

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It is well-known that the appearance of almost-empty (shallow) conduction bands in solids strongly affects their superconducting properties. In a shallow band charge carriers are depleted and have nearly zero velocities so that the crossover from the Bardeen-Cooper-Schrieffer (BCS) superfluidity to Bose-Einstein condensation (BEC) is approached. Based on a two-band prototype system with one shallow and one deep band, we demonstrate that the fundamental phase diagram of the superconducting magnetic response changes qualitatively as compared to standard superconductors with only deep bands. The so-called intertype (IT) domain between superconductivity types I and II systematically expands in the phase diagram when passing from the BCS to BEC side: its width is inversely proportional to the squared Cooper-pair radius that shrinks several orders of magnitude through the crossover. We also show that the coupling to a stable condensate of the deep band makes the system rather robust against the otherwise strong superconducting fluctuations. Thus, the BCS-BEC crossover induced by a shallow band pushes standard superconductivity types wide apart so that the IT domain tends to dominate the phase diagram and therefore the magnetic properties of shallow-band superconductors.

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The crossover from the Bardeen-Cooper-Schrieffer (BCS) superfluidity to Bose-Einstein condensation (BEC) is usually investigated in trapped ultracold fermionic gases with the resonant scattering [1–4]. However, it was originally proposed for solids with a shallow conduction band whose lower edge is close to the chemical potential [5], see also the review in [6]. Although theoretical studies of the BCS-BEC crossover in superconductors have a long history, its unambiguous experimental evidences have been obtained only recently in $\text{FeSe}_x\text{Te}_{1-x}$, where shallow Fermi pockets were proved to play a significant role [7–9]. Interest in such superconducting materials is fueled by expectations of a higher critical temperature T_c [10, 11] and novel multigap/multicondensate coherent phenomena [12] potentially useful for technological applications. Here we demonstrate that the BCS-BEC crossover regime realized in a multiband superconductor can profoundly influence the superconducting magnetic properties so that the relevant phase diagram differs strikingly from the standard one.

Earlier investigations of the BCS-BEC crossover in a charged superfluid considered a sharp interchange between types I and II throughout the entire crossover interval [13]. This consideration was based on the Ginzburg-Landau (GL) theory according to which the types I and II interchange when the GL parameter $\kappa = \lambda/\xi$ (λ and ξ are the magnetic and coherence lengths) crosses the critical value $\kappa_0 = 1/\sqrt{2}$ [14–16]. However, it is known that

the results of the GL theory for the phase diagram of the superconducting magnetic properties are valid only in the limit $T \rightarrow T_c$. In particular, below T_c the intertype (IT) regime is not reduced to the single point $\kappa = \kappa_0$ but occupies a finite temperature-dependent interval of κ 's, forming the IT domain in the (κ, T) plane [17–26]. Magnetic properties of a superconductor in this domain are governed by the Bogomolnyi duality between the magnetic field and the condensate density (self-duality) [25, 26, 28, 29], that facilitates formation of exotic flux/condensate configurations such as, e.g., a lattice of superconducting islands, stripe/labyrinths patterns, giant vortices, and vortex clusters [27]. In conventional single-band superconductors the IT domain is almost negligible and thus ignored in textbooks. However, in multiband superconducting materials it shows a general tendency to increase due to enhancement of the non-local effects [26].

This work demonstrates that the proximity to the BCS-BEC crossover induced by a shallow band has a dramatic effect on the IT domain: its width is inversely proportional to the squared Cooper-pair radius which shrinks several orders of magnitude when passing from the BCS to BEC regime [30–32]. Our analysis is done for a two-band prototype system with one shallow and one deep band, where closed analytical results can be derived. However, a universal character of the employed formalism allows one to expect qualitatively similar results for sys-

tems with an arbitrary number of bands. Our conclusions are obtained within the mean-field theory, which remains valid despite the presence of a shallow band: coupling to the stable condensate of the deep band screens the otherwise strong superconducting fluctuations [33].

The IT domain is described by the critical GL parameters κ_i^* , each marking the appearance of a particular flux configuration i at the thermodynamic critical field H_c : at $\kappa > \kappa_i^*$ configuration i becomes more favorable energetically than the Meissner state [18–21, 25, 26]. The critical GL parameters are found from the equation

$$\mathfrak{G}(\kappa_i^*, T) = 0, \quad \mathfrak{G} = \int d\mathbf{r} \left(\mathfrak{f} + \frac{H_c^2}{8\pi} - \frac{H_c B}{4\pi} \right), \quad (1)$$

where \mathfrak{G} is the difference between the Gibbs free-energy of configuration i and of the Meissner state at H_c , and \mathfrak{f} is the condensate free-energy density. The magnetic induction \mathbf{B} is assumed to be parallel to the external field \mathbf{H} . Equation (1) can also be used to find a critical GL parameter associated with qualitative changes in some properties of a mixed state, e.g., in the sign of the long-range vortex-vortex interaction [21, 22]. Due to the Bogomolnyi self-duality, the GL theory is infinitely degenerate at $\kappa = \kappa_0$ and $H = H_c$ [25, 26, 28, 29]. Then it predicts that all critical GL parameters are equal to κ_0 [25, 26, 34]. However, when corrections to the GL theory are taken into account, the degeneracy is removed, giving rise to an infinite number of different temperature dependent κ_i^* , that shape the IT domain in the (κ, T) plane. Its lower boundary κ_{min}^* is found from the onset of the superconductivity nucleation at H_c , which is equivalent to the condition $H_{c2} = H_c$, with H_{c2} the upper critical field. The upper boundary κ_{max}^* is determined by the appearance of the long-range attractive Abrikosov vortices. At $T \rightarrow T_c$ the GL theory is exact and thus in this limit $\kappa_i^* \rightarrow \kappa_0$ for all i .

We calculate κ_i^* using the extended GL (EGL) formalism that incorporates the leading corrections to the GL theory within the perturbation expansion of the BCS equations over $\tau = 1 - T/T_c$ [35–37]. Here a sketch of the derivation is presented where we highlight differences with the earlier works [26, 36] that appear due to the presence of the shallow band. The BCS free-energy density for a two-band system writes as

$$\mathfrak{f} = \frac{\mathbf{B}^2}{8\pi} + \Delta^\dagger \tilde{g}^{-1} \Delta + \sum_{\nu=1,2} \mathfrak{f}_\nu, \quad (2)$$

where the vector $\Delta^\dagger = (\Delta_1^*, \Delta_2^*)$ comprises the band gap functions Δ_ν , \tilde{g}^{-1} is the inverse of the coupling matrix \tilde{g} ($g_{ij} = g_{ji}$ are real) and \mathfrak{f}_ν is a functional of Δ_ν . Expand-

ing \mathfrak{f}_ν in powers of Δ_ν and its gradients, one finds [26, 36]

$$\begin{aligned} \mathfrak{f}_\nu = & -a_{1,\nu} |\Delta_\nu|^2 + a_{2,\nu} |\mathbf{D}\Delta_\nu|^2 - a_{3,\nu} \left(|\mathbf{D}^2\Delta_\nu|^2 \right. \\ & + \frac{\text{rot}\mathbf{B} \cdot \mathbf{i}_\nu}{3} + \frac{4e^2}{\hbar^2 c^2} \mathbf{B}^2 |\Delta_\nu|^2 \Big) + a_{4,\nu} \mathbf{B}^2 |\Delta_\nu|^2 \\ & + \frac{b_{1,\nu}}{2} |\Delta_\nu|^4 - \frac{b_{2,\nu}}{2} \left(L_\nu |\Delta_\nu|^2 |\mathbf{D}\Delta_\nu|^2 \right. \\ & \left. + l_\nu [(\Delta_\nu^*)^2 (\mathbf{D}\Delta_\nu)^2 + \text{c.c.}] \right) - \frac{c_{1,\nu}}{3} |\Delta_\nu|^6, \end{aligned} \quad (3)$$

where $a_{n,\nu}$, $b_{n,\nu}$, $c_{n,\nu}$ are temperature-dependent band coefficients, L_ν and l_ν are constants introduced here to capture qualitative differences (e.g., dimensionality, depth, etc.) between the contributing bands (compare Eq. (3) with Eq. (27) in [36]) and

$$\mathbf{i}_\nu = -\frac{4e}{\hbar c} \text{Im}[\Delta_\nu \mathbf{D}^* \Delta_\nu^*], \quad \mathbf{D} = \nabla - \mathbf{i} \frac{2e}{\hbar c} \mathbf{A}. \quad (4)$$

The τ -expansion is obtained by representing all entering quantities as τ -series

$$\begin{aligned} \Delta &= \tau^{1/2} [\Delta^{(0)} + \tau \Delta^{(1)}], \quad \mathbf{B} = \tau [\mathbf{B}^{(0)} + \tau \mathbf{B}^{(1)}], \\ \mathbf{A} &= \tau^{1/2} [\mathbf{A}^{(0)} + \tau \mathbf{A}^{(1)}], \quad H_c = \tau [H_c^{(0)} + \tau H_c^{(1)}], \end{aligned} \quad (5)$$

where the lowest-order (GL) contributions and leading corrections are given. We also invoke the τ -scaling of the coordinates [36, 37] which leads to the additional factor $\tau^{1/2}$ for each gradient in the τ -expansion of the functional. The temperature-dependent band coefficients in Eq. (3) are obtained as

$$\begin{aligned} a_{1,\nu} &= \mathcal{A}_\nu - \tau [a_\nu^{(0)} + \tau a_\nu^{(1)}], \quad a_{2,\nu} = \mathcal{K}_\nu^{(0)} + \tau \mathcal{K}_\nu^{(1)}, \\ a_{3,\nu} &= \mathcal{Q}_\nu^{(0)}, \quad a_{4,\nu} = r_\nu^{(0)}, \quad b_{1,\nu} = b_\nu^{(0)} + \tau b_\nu^{(1)}, \\ b_{2,\nu} L_\nu &= \mathcal{L}_\nu^{(0)}, \quad b_{2,\nu} l_\nu = \ell_\nu^{(0)}, \quad c_{1,\nu} = c_\nu^{(0)}, \end{aligned} \quad (6)$$

where the τ -expansion coefficients are calculated using a particular microscopic model of the band. Substituting Eqs. (5), (6) and the gradient scaling into Eqs. (2) and (3), one obtains the τ -expansion for the free-energy density. It is then inserted in Eq. (1), which gives the corresponding series for \mathfrak{G} .

To obtain the two lowest orders in the series for \mathfrak{G} , one needs only $\Delta_{1,2}^{(0)}$ and $\mathbf{B}^{(0)}$ because the contributions containing $\Delta_\nu^{(1)}$ and $\mathbf{B}^{(1)}$ are either zero (up to the vanishing surface integrals) or can be rewritten through $\Delta_{1,2}^{(0)}$ and $\mathbf{B}^{(0)}$ [26]. As a result, the two leading contributions to \mathfrak{G} can be calculated using only the solution to the GL formalism. The latter takes the form

$$\begin{pmatrix} \Delta_1^{(0)} \\ \Delta_2^{(0)} \end{pmatrix} = \Psi(\mathbf{r}) \begin{pmatrix} S^{-1/2} \\ S^{1/2} \end{pmatrix}, \quad (7)$$

where Ψ is the Landau order parameter that satisfies the single-component GL equation, and S is determined by the linearized gap equation that yields

$$S = \frac{1}{g_{12}} (g_{22} - G\mathcal{A}_1) = \frac{g_{12}}{g_{11} - G\mathcal{A}_2}, \quad (8)$$

ν	$\mathcal{M}_{b,\nu}^{(0)}$	$\mathcal{M}_{c,\nu}^{(0)}$	$\mathcal{M}_{\mathcal{K},\nu}^{(0)}$	$\mathcal{M}_{\mathcal{Q},\nu}^{(0)}$	$\mathcal{M}_{\mathcal{L},\nu}^{(0)}$	$\mathcal{M}_{a,\nu}^{(1)}$	$\mathcal{M}_{b,\nu}^{(1)}$	$\mathcal{M}_{\mathcal{K},\nu}^{(1)}$
1	$7\zeta(3)/(8\pi^2)$	$93\zeta(5)/(128\pi^4)$	$7\zeta(3)/(32\pi^2)$	$93\zeta(5)/(2048\pi^4)$	$31\zeta(5)/(32\pi^4)$	1/2	2	2
2	$7\zeta(3)/(8\pi^2)$	$93\zeta(5)/(128\pi^4)$	$3\zeta(2)/(8\pi^2)$	$7\zeta(3)/(512\pi^2)$	$25\zeta(4)/(16\pi^4)$	1/2	2	1

TABLE I. Numerical factors $\mathcal{M}_{w,\nu}^{(0)}$ (with $w = b, c, \mathcal{K}, \mathcal{Q}, \mathcal{L}$) and $\mathcal{M}_{w,\nu}^{(1)}$ (with $w = a, b, \mathcal{K}$) for the deep ($\nu = 1$) and shallow ($\nu = 2$) bands, see Eqs. (13) and (14). In the table $\zeta(x)$ is the Riemann zeta function of x .

with $G = \det \check{g} = g_{11}g_{22} - g_{12}^2$.

The critical GL parameters are sought in the form $\kappa^* = \kappa_0 + \delta\kappa$, where $\delta\kappa \sim \tau$ (we hide the index i unless it causes confusion). Then, we utilize a self-dual form of the GL theory (the self-duality Bogomolnyi equations) [28, 29] which allows one to explicitly express $\mathbf{B}^{(0)}$ as a function of $|\Psi|^2$. The resulting expansion of \mathfrak{G} contains only the linear terms $\propto \tau$ and $\propto \delta\kappa$, while the GL contribution vanishes due to the degeneracy of the GL theory at κ_0 and H_c . Resolving Eq. (1), one obtains $\kappa^* = \kappa_0 + \tau\kappa^{*(1)}$ with

$$\frac{\kappa^{*(1)}}{\kappa_0} = \bar{\mathcal{K}} - \bar{c} + 2\bar{\mathcal{Q}} + \bar{G}\bar{\beta}(2\bar{\alpha} - \bar{\beta}) + \frac{\mathcal{I}}{\mathcal{I}} \left(\frac{\bar{\mathcal{L}}}{4} - \bar{c} - \frac{5}{3}\bar{\mathcal{Q}} - \bar{G}\bar{\beta}^2 \right), \quad (9)$$

where the dependence on a particular mixed-state configuration enters via the integrals

$$\mathcal{I} = \int |\Psi|^2 (1 - |\Psi|^2) d\mathbf{r}, \quad \mathcal{J} = \int |\Psi|^4 (1 - |\Psi|^2) d\mathbf{r}. \quad (10)$$

Dimensionless constants in Eq. (9) are given by

$$\begin{aligned} \bar{\mathcal{K}} &= \frac{\mathcal{K}^{(1)}}{\mathcal{K}} - \frac{b^{(1)}}{2b}, \quad \bar{c} = \frac{ca}{3b^2}, \quad \bar{\mathcal{Q}} = \frac{a\mathcal{Q}}{\mathcal{K}^2}, \quad \bar{\mathcal{L}} = \frac{a\mathcal{L}}{b\mathcal{K}}, \\ \bar{G} &= \frac{Ga}{4g_{12}}, \quad \bar{\alpha} = \frac{\alpha}{a} - \frac{\Gamma}{\mathcal{K}}, \quad \bar{\beta} = \frac{\beta}{b} - \frac{\Gamma}{\mathcal{K}}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} w &= w_1^{(0)} S^{-p} + S^p w_2^{(0)}, \quad w^{(1)} = w_1^{(1)} S^{-p} + S^p w_2^{(1)}, \\ \alpha &= a_1^{(0)} S^{-1} - S a_2^{(0)}, \quad \beta = b_1^{(0)} S^{-2} - S^2 b_2^{(0)}, \\ \Gamma &= \mathcal{K}_1^{(0)} S^{-1} - S \mathcal{K}_2^{(0)}. \end{aligned} \quad (12)$$

with the substitutions $w = \{a, \mathcal{K}, \mathcal{Q}, b, \mathcal{L}, c\}$, $w_\nu^{(0)} = \{a_\nu^{(0)}, \mathcal{K}_\nu^{(0)}, \mathcal{Q}_\nu^{(0)}, b_\nu^{(0)}, \mathcal{L}_\nu^{(0)}, c_\nu^{(0)}\}$, $w^{(1)} = \{\mathcal{K}^{(1)}, b^{(1)}\}$, and $w_\nu^{(1)} = \{\mathcal{K}_\nu^{(1)}, b_\nu^{(1)}\}$. In Eq. (12) $p = 1$ appears in w 's related to $a_{n,\nu}$ while $p = 2$ and 3 correspond to $b_{n,\nu}$, and $c_{n,\nu}$, respectively. Notice that the terms containing $a_\nu^{(1)}$ and $\ell_\nu^{(0)}$ do not contribute to \mathfrak{G} and, thus, to Eq. (9). In turn, $r_\nu^{(0)}$ is negligible and so ignored. One notes that Eqs. (9)-(12) differ from the results obtained for a system with two deep bands in Ref. [26]: here the expression for κ^* contains additional terms.

Coefficients in Eq. (6) are calculated assuming that both bands have a 2D circular-symmetry Fermi surface with the band single-particle dispersion $\varepsilon_{\nu,k} = \varepsilon_{\nu,0} + (\hbar^2/2m_\nu)(k_x^2 + k_y^2)$, where $\varepsilon_{\nu,0}$ is the band lower edge and m_ν is the band carrier mass. The magnetic field is chosen in the z direction to deal with the isotropic system. The calculations are performed in the clean limit. For the deep band ($\nu = 1$), one employs the standard BCS approximations [14–16] as $\Delta_1 \ll \mu - \varepsilon_{1,0}$. Calculations for the shallow band ($\nu = 2$) are technically more involved. However, analytic expressions for all coefficients can be derived when the chemical potential touches the band lower edge, i.e., $\mu = \varepsilon_{2,0}$ (this can be assumed without the generality loss). Notice, that the 2D character of the shallow band is important: although the band is almost empty, its DOS at the lower edge remains sizeable. Calculations yield the following lowest-order coefficients in Eq. (6):

$$\begin{aligned} \mathcal{A}_\nu &= N_\nu \ln \left(\frac{2e^\gamma \hbar \omega_c}{\pi T_c} \right), \quad a_\nu^{(0)} = -N_\nu, \quad b_\nu^{(0)} = N_\nu \frac{\mathcal{M}_{b,\nu}^{(0)}}{T_c^2}, \\ c_\nu^{(0)} &= N_\nu \frac{\mathcal{M}_{c,\nu}^{(0)}}{T_c^4}, \quad \mathcal{K}_\nu^{(0)} = N_\nu \mathcal{M}_{\mathcal{K},\nu}^{(0)} \frac{\hbar^2 v_\nu^2}{T_c^2}, \\ \mathcal{Q}_\nu^{(0)} &= N_\nu \mathcal{M}_{\mathcal{Q},\nu}^{(0)} \frac{\hbar^4 v_\nu^4}{T_c^4}, \quad \mathcal{L}_\nu^{(0)} = N_\nu \mathcal{M}_{\mathcal{L},\nu}^{(0)} \frac{\hbar^2 v_\nu^2}{T_c^4}, \end{aligned} \quad (13)$$

where $\hbar \omega_c$ is the cut-off energy, γ is the Euler constant, N_ν is the band DOS, and v_ν denotes the characteristic band velocity, i.e., the Fermi velocity $v_F = \sqrt{2(\mu - \varepsilon_{1,0})/m_1}$ for the deep band and the temperature-driven velocity $v_T = \sqrt{2T_c/m_2}$ for the shallow band. The factors $\mathcal{M}_{w,\nu}^{(0)}$ are given in Tab. I. The band DOS are $N_1 = \tilde{N}_1 m_1 / (2\pi \hbar^2)$ and $N_2 = \tilde{N}_2 m_2 / (4\pi \hbar^2)$, with \tilde{N}_ν introduced to account for the states in the z direction (this quantity does not affect final conclusions). The next-order coefficients in Eq. (6) are obtained as

$$w_\nu^{(1)} = \mathcal{M}_{w,\nu}^{(1)} w_\nu^{(0)}, \quad (14)$$

where $w = \{a, \mathcal{K}, b\}$ and $\mathcal{M}_{w,\nu}^{(1)}$ are also shown in Tab. I.

We can now calculate the boundaries of the IT domain. κ_{min}^* is obtained from the condition that the inhomogeneous mixed state disappears at H_c , which gives $\mathcal{J}/\mathcal{I} = 0$ (as $\Psi \rightarrow 0$). At $\kappa = \kappa_{max}^*$ the long-range vortex-vortex interaction changes its sign. For the two-vortex

solution one finds $\mathcal{J}(R)/\mathcal{I}(R) \rightarrow 2$ in the limit of large inter-vortex distance $R \rightarrow \infty$ [26]. Using these results, one obtains final expressions for the critical GL parameters κ_{min}^* and κ_{max}^* as complicated algebraic functions of the microscopic parameters $\eta = N_2/N_1$, v_2/v_1 , and g_{ij} . Obtained expressions, however, can be simplified because $v_2/v_1 \sim \sqrt{T_c/\mu} \ll 1$ and $\eta \gg 1$ (the latter inequality is dictated by the proximity to the BCS-BEC crossover [31, 32]). In this case the difference $\delta\kappa^* = \kappa_{max}^* - \kappa_{min}^*$ reduces to

$$\frac{\delta\kappa^*}{\kappa_0\tau} \simeq -\frac{10}{3}\bar{Q} \simeq 2.27 \left(\frac{\lambda_{22}}{\lambda_{12}}\right)^2 \eta, \quad (15)$$

where $\bar{Q} \simeq [93\zeta(5)/98\zeta^2(3)]S^2\eta$ and S approaches $\lambda_{22}/\lambda_{12}$, with the dimensionless coupling $\lambda_{ij} = g_{ij}N$ ($N = N_1 + N_2$). The expression for S follows from the solution to Eq. (8), i.e.,

$$S = \frac{1}{2\lambda_{12}} \left[\lambda_{22} - \frac{\lambda_{11}}{\eta} + \sqrt{\left(\lambda_{22} - \frac{\lambda_{11}}{\eta}\right)^2 + 4\frac{\lambda_{12}^2}{\eta}} \right]. \quad (16)$$

Equation (15) demonstrates that the IT domain systematically enlarges when approaching the BCS-BEC crossover. This enlargement is more pronounced when increasing $\lambda_{22}/\lambda_{12}$, i.e., when the role of the deep band is further diminished. It is important to note that the dominant contribution to the enlargement given by Eq. (15) is provided by the most non-local terms in the free energy functional, i.e., those with the fourth order gradients, see Eq. (3). This is closely connected to the known fact that the IT domain appears due to non-local interactions in the condensate beyond the standard GL theory [38]. Numerical results for $\delta\kappa^*$ given in Fig. 1(a) as a function of η , are calculated using the original not simplified expressions at $v_2/v_1 = 0$, where we set $\lambda_{11} = \lambda_{22} = 0.3$ and $\lambda_{22}/\lambda_{12} = 1, 2$ and 3 (results are qualitatively similar for any choice of the coupling constants). One can see that $\delta\kappa^*$ exhibits a linear dependence of the simplified Eq. (15) already at moderate values of $\eta \gtrsim 1$.

Finally we address the role of the superconducting fluctuations. It is commonly expected that the fluctuations are enhanced in superconductors with shallow bands, which can compromise the mean-field results. However, in multiband systems even a weak coupling to a deep band can screen the fluctuations [33]. We estimate their relative importance by calculating the Ginzburg number Gi , i.e., the temperature interval near T_c where the fluctuation-induced contribution to the heat capacity δc_V exceeds its mean-field counterpart $c_{0,V}$ [16]. For bands with 2D Fermi surfaces we get

$$c_{0,V} = \frac{a^2}{bT_c}, \quad \delta c_V \simeq \frac{1}{4\pi\xi_0^2 d_z \tau}, \quad (17)$$

where a, b are given by Eq. (12), and $d_z/2\pi$ is the inverse of the size of the Brillouine zone in the z direction.

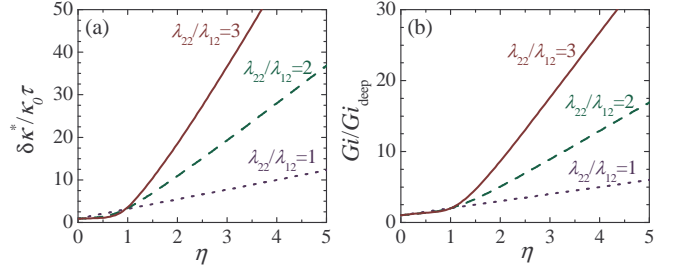


FIG. 1. (a) The size of the intertype domain (relative to $\kappa_0\tau$) as a function of η calculated at $v_2/v_1 = 0$ and $\lambda_{11} = \lambda_{22} = 0.3$ for $\lambda_{22}/\lambda_{12} = 1$ (dotted), 2 (dashed), and 3 (solid). (b) The relative Ginzburg number Gi/Gi_{deep} versus η obtained for the same parameters as in panel (a).

The Cooper-pair radius (the BCS coherence length) ξ_0 is found for the two-band system as

$$\xi_0^2 = -\frac{\mathcal{K}}{a} = \frac{\xi_{0,1}^2}{1 + \eta S^2} + \frac{\xi_{0,2}^2}{1 + \eta^{-1} S^{-2}}, \quad (18)$$

where $\xi_{0,\nu}^2 = -\mathcal{K}_\nu^{(0)}/a_\nu^{(0)}$ are the band BCS lengths. Taking into account that ξ_0 approaches $\xi_{2,0} \approx 0$ at $\eta \rightarrow \infty$, one can see that in this limit δc_V sharply increases. Still, unless η assumes extreme values, the contribution of the deep band to ξ_0 remains dominant even at very large η as long as $\xi_{0,1} \gg \xi_{0,2}$. As a result ξ_0^2 and δc_V remain close to their deep-band values, which is seen as the screening of the superconducting fluctuations [33]. Gi is obtained by resolving $c_{0,V} = \delta c_V|_{\tau=Gi}$,

$$Gi = Gi_{deep} \frac{(1 + \eta)(1 + \eta S^4)}{1 + \eta S^2}, \quad (19)$$

where Gi_{deep} is calculated for the deep band with the DOS $N = N_1 + N_2$. For $v_2/v_1 = 0$ and $\eta \rightarrow \infty$, Eq. (19) yields $Gi/Gi_{deep} \simeq \eta(\lambda_{22}/\lambda_{12})^2$. Comparing Eqs. (15), (18) and (19), we arrive at

$$\delta\kappa^*/\kappa_0 \propto Gi/Gi_{deep} \propto (k_F \xi_0)^{-2}, \quad (20)$$

which relates the width of the IT domain and the Gi parameter with the Cooper-pair radius given in units of k_F^{-1} (k_F is the Fermi wavenumber in the deep band). We note that it is a comparison of ξ_0 with the average inter-particle distance that defines the proximity to the BCS-BEC crossover, and in the case of interest this distance is of the order of k_F^{-1} . Taking into account that for typical superconducting systems $Gi_{deep} \sim 10^{-16} \div 10^{-6}$ [16], one concludes that the presence of a shallow band does not invalidate the mean-field approach in a very wide parameter range: the fluctuations can be neglected even if ξ_0 drops by several orders of magnitude. For illustration Fig. 1(b) shows Gi/Gi_{deep} as a function of η , calculated for the same parameters as the results for $\delta\kappa^*$

in panel (a). One sees that, similarly to $\delta\kappa^*$, Gi quickly approaches the linear asymptotic regime at $\eta \gtrsim 1$.

In summary, we have demonstrated that the BCS-BEC crossover, induced by the presence of a shallow band, pushes apart standard superconductivity types I and II by enhancing the IT domain of unconventional superconducting magnetic properties. This effect is mainly controlled by the highest-gradient contribution to the free energy functional. Although our analysis focuses on the two-band model, we expect that our conclusions hold for a general multiband superconductor with at least one shallow and one deep band in the electronic spectrum. The reason is that the EGL formalism yields formally similar results for an arbitrary number of bands, as long as there is no additional symmetry in the system, see [36]. We note that our choice of the dimensionality for the deep band is not crucial and qualitatively similar results can be obtained for the deep band with a 3D Fermi surface. On the contrary, when assuming a 3D Fermi surface for the shallow band, the BCS-BEC crossover can be reached only for an abnormally strong coupling due to negligibly small DOS [39]. Finally, the calculations of the Ginzburg number have confirmed that the presence of a shallow band(s) in multiband superconductors does not compromise the mean-field approach in a wide parameter range because of the screening effects generated by the coupling to deep bands.

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